



Addressing misconceptions regarding seismic hazard assessment in mines: b -value, M_{max} , and space-time normalization

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Synopsis

Seismic hazard assessment, in some form or another, has formed part of seismic risk management in seismically active hard-rock mines for decades. Some misconceptions, however, exist in the mining industry which may lead to errors in interpretation and poor risk management decisions. This paper addresses some misconceptions the author has encountered in the mining industry. This is done by exploring the meaning and implications of the frequency–magnitude distribution. The meaning of M_{max} , the methods of assessing it, and the topic of space and time normalization necessary for the evaluation of seismic hazard, are also addressed. The scope of this paper does not include the evaluation of strong ground motion exceedance which also forms part of the evaluation of seismic hazard at mine sites.

Keywords

Seismic risk, frequency–magnitude distribution, normalization probability.

Introduction

At any seismically active mine, considerable effort is invested into the effective management of seismic risk (see Potvin *et al.*, 2019), of which seismic hazard assessment is, of course, a fundamental component. Over many years I have come across several misconceptions regarding the assessment of the seismic hazard which adversely affect the standard of seismic risk management in our industry. Some of these misconceptions are widespread and deeply rooted, while others crop up from time to time and seem to migrate through the industry. The aim of this paper is to address some of those misconceptions.

The frequency–magnitude (FM) distribution is foundational to understanding seismic hazard. It appears, however, that many of the misunderstandings regarding seismic hazard arise from inadequate understanding of the meaning and implications of the FM distribution. For this reason, a large proportion of this paper will be devoted to the FM distribution and its implications. The fact that ' M_{max} ' is used for several different concepts, further creates confusion and misunderstanding. Another topic that does not seem to be well understood is that of normalization of hazard, with respect to space and time, and the related issue of separating sources of seismicity with different behaviour.

The assessment of seismic hazard in mines also includes the evaluation of strong ground motions at specific locations. This topic is, however, excluded from the scope of this paper.

What is the Gutenberg–Richter relationship really?

It appears that many misconceptions in the mining industry stem from an inadequate understanding of the FM distribution and its implications. Many rock engineers use the FM chart without realizing that it is simply a reverse cumulative distribution¹ of magnitude, with the vertical axis plotted on a log scale. The straight line fit, or any other curve fitted to the data, is therefore simply a statistical best-fit model and is conceptually the same as, for example, a normal distribution fitted to UCS data (see Figure 1).

¹Also referred to as Complementary Cumulative Distribution or Inverse Cumulative Distribution. The term 'Inverse Cumulative Distribution', however, is also used to refer to the quantile function.

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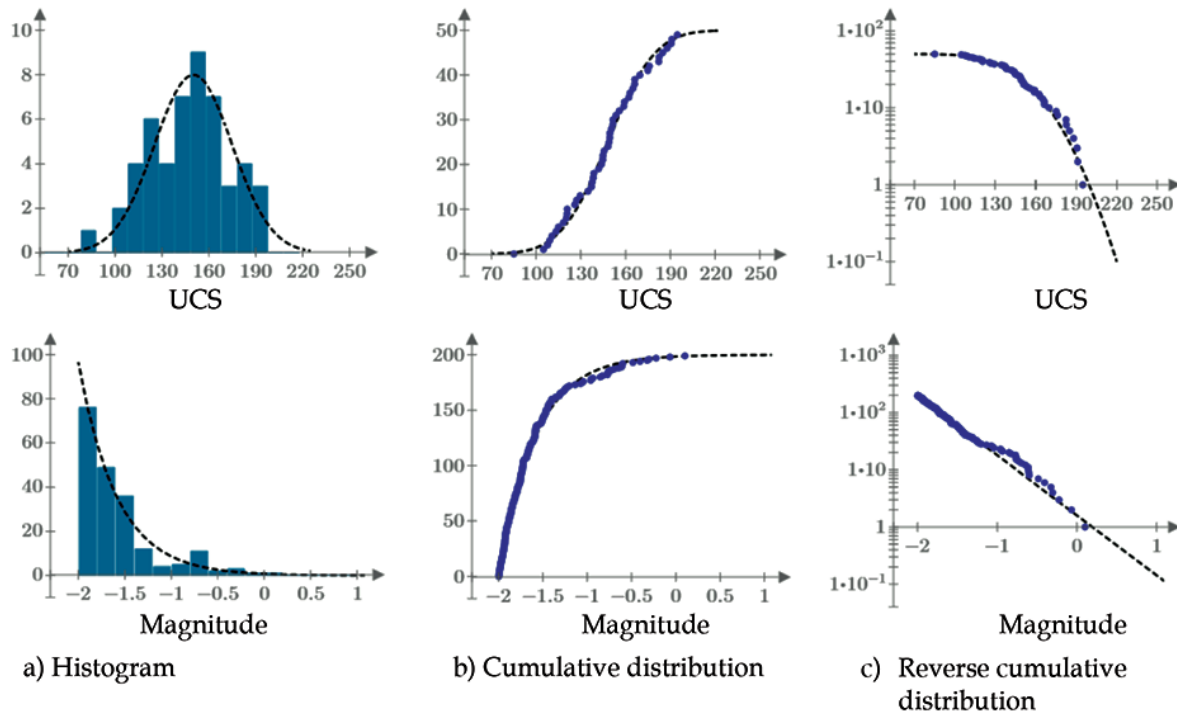


Figure 1—Graphical representation and the statistical best-fit model of UCS data and seismic magnitude data

The model most often used for seismic magnitude distribution is the Gutenberg–Richter (GR) relationship (Gutenberg and Richter, 1944):

$$\log(N) = a - b \cdot M \quad [1]$$

Which we can write as:

$$N = 10^{a-b \cdot M} \quad [2]$$

where:

- N is the number of events with magnitude $\geq M$
- M is the event magnitude
- a is the coefficient quantifying the number of events
- b is the coefficient quantifying the log relative occurrence of smaller to larger events

In the case of seismic magnitude data, however, limitations in the system sensitivity result in small events not being recorded. For statistical and probabilistic analysis, one can therefore only use the data above the magnitude of completeness, m_{min} , with the statistical density and cumulative function being limited to the magnitude ranges greater than m_{min} . We can write the GR relationship in Equation [2], with respect to the reverse cumulative distribution function, as follows:

$$N = N_{mmin} \cdot F'(M) \quad [3]$$

with

$$F'(M) = \begin{cases} 1, & M < m_{min} \\ 10^{-b(M-m_{min})}, & M \geq m_{min} \end{cases}$$

and

$$N_{mmin} = 10^{a-b \cdot m_{min}}$$

Where

N_{mmin} is the number of events with magnitude greater than m_{min}

$F'(M)$ is the reverse cumulative distribution function

It is interesting to note that $F'(M)$ in Equation 2 is simply a different formulation of the commonly used negative exponential distribution, translated to start at m_{min} instead of zero.

The open-ended GR relationship (Equations [1] and [2]) predicts a non-zero probability of physically impossible magnitude sizes, (*i.e.*, the probability of a Richter magnitude > 10 is not zero!). For this reason, equations that truncate at large magnitudes are preferred, for which several different relationships have been proposed (Utsu, 1999). In mining, the truncated Gutenberg–Richter (TGR) relationship (Page, 1968) is often used. Equation [4] rewrites Page's formulation to follow the general formulation that is commonly used by rock engineers, as presented in Equation [3].

$$N = N_{mmin} \cdot F'(M)$$

with the reverse cumulative distribution written as:

$$F'(M) = \begin{cases} 1, & M < m_{min} \\ 1 - \frac{1-10^{-b(M-m_{min})}}{1-10^{-b(M_{UL}-m_{min})}}, & m_{min} \leq M < M_{UL} \\ 0, & M \geq M_{UL} \end{cases} \quad [4]$$

where

N_{mmin} is the number of events with magnitude greater than m_{min}

M_{UL} is the upper truncation magnitude of the distribution.

Similar to Equation [2], Equation [4] is a different formulation of a truncated and translated negative exponential distribution.

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The cumulative distribution function, $F(M)$ and the probability density function, $f(M)$ for the TGR can be written as follows:

$$F(M) = \begin{cases} 0, & M < m_{min} \\ \frac{1-10^{-b(M-m_{min})}}{1-10^{-b(M_{UL}-m_{min})}}, & m_{min} \leq M < M_{UL} \\ 1, & M \geq M_{UL} \end{cases} \quad [5]$$

$$f(M) = \begin{cases} 0, & M < m_{min} \\ \frac{\ln(10) \cdot b \cdot 10^{-b(M-m_{min})}}{1-10^{-b(M_{UL}-m_{min})}}, & m_{min} \leq M < M_{UL} \\ 0, & M \geq M_{UL} \end{cases} \quad [6]$$

The M_{max} confusion

In the mining industry, the term ' M_{max} ' is used for several different concepts, with the loose meaning of 'expected largest event' associated with it. These concepts are (see Figure 2):

- the largest recorded event in a dataset (X_{max})
- the value of the fitted GR relationship at $N=1$ (a/b)
- the distribution of the largest expected magnitude (f_{max})
- the largest physically possible event (M_{max})
- the upper truncation of the FM distribution (M_{UL}).

Each of these is, in some way or another, being used as hazard indicators and they are discussed in the following paragraphs. Generally, no distinction is made between M_{max} and M_{UL} , and the two concepts are mostly used interchangeably. Conceptually M_{max} has a physical meaning, while M_{UL} carries only the abstract meaning of being an upper limit of a probability distribution. I find it useful to keep these two concepts separated and will discuss this in more detail further on in the paper.

Due to the stochastic nature of seismicity, the largest event magnitude within a given number of events is a distribution and cannot be captured with a single number. This distribution is represented as a colour spectrum at $N=1$ in Figure 2. In this paper, the probability density, cumulative probability and reverse cumulative probability functions of the distribution are respectively referred to as f_{max} , F_{max} and F'_{max} .

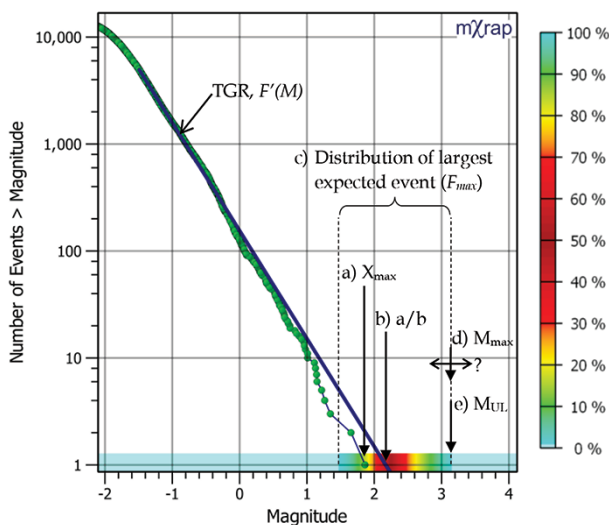


Figure 2—FM distribution of seismic magnitude

It is important to note that any statement regarding the expectation of specified event magnitudes is dependent on the number of events. For any such statement to be meaningful, it is necessary to specify the number of events applicable to this statement. In this paper, the short notation, $N@M$, is used; for example, $N@-1=100$ or $N@0 = 1$, and the a -value in Equations [1] to [4] is equal to $\log(N@0)$.

Distribution of the expected largest event

The two parameters a/b and M_{UL} , shown in Figure 2 are important parameters in the assessment and communication of seismic hazard (Kijko and Funk, 1994; Jager and Ryder, 1999; Hudyma and Potvin, 2004; Hudyma, 2008; Mendecki, 2008; Hudyma and Potvin, 2010; Mendecki, 2012; Mendecki, 2013b). It appears though, that undue emphasis is sometimes placed on these parameters, which do not fully capture the seismic hazard. Consider the distribution of the maximum expected event within $N@-1.5 = 4000$, shown in Figure 2; the value of M_{UL} defines only the upper limit of the $F(M)$ and $F_{max}(M)$ distributions and the a/b -value describes only one point on those distributions. The whole body of the distribution, however, describes the hazard.

The distribution of the expected largest event can be obtained directly from the FM distribution. The cumulative distribution function of the largest magnitude within n events is given by (Gibowicz and Kijko, 1994):

$$F_{max}(M, n) = (F(M))^n \quad [7]$$

where

$F_{max}(M, n)$ is the cumulative distribution function of the magnitude of the largest event within n events
 n is the the number of events with magnitude $\geq m_{min}$

The probability density function of the largest event within n events can be obtained as the derivative of F_{max} and is as follows:

$$f_{max}(M, n) = \begin{cases} 0, & M < m_{min} \\ n \cdot f(M) \cdot F(M)^{n-1}, & m_{min} \leq M < M_{UL} \\ 0, & M \geq M_{UL} \end{cases} \quad [8]$$

where

$f_{max}(M, n)$ is the probability density functions of the magnitude of the largest event within n events

$F(M), f(M)$ is the the cumulative distribution and probability density functions of the FM distribution (e.g. Equation 5 and 6)

n is the the number of events with magnitude $\geq m_{min}$

It is well-known that the intensity of dynamic waves attenuates sharply with distance. Using the Canadian Rockburst Handbook formulation (Mining Research Directorate, 1996), one can estimate that the body wave peak particle velocity (ppv) resulting from a Richter magnitude (M_L) of 3 at a distance of 500 m is similar to that from a M_L 0.5 at a distance of 20 m. Assuming a b -value of 1, one would expect about 315 events of $M_L > 0.5$ for every single event of $M_L > 3$. When considering the fact that the workforce is exposed to the occurrence of many more small events at close proximity, it is clear that the body of the distribution, and not only the upper limit or the mode of the distribution, is important. This is evidenced in the fact that it is becoming more common to install face support in development headings to protect the workforce against the effect of events much smaller than the M_{UL} .

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The consequence of smaller events is, of course, expected to be much smaller than that of an M_{max} event. One useful way to quantify the hazard, therefore, is to express it as the probability of exceeding a large event, for example, M_L1 , M_L2 , and $M_L2.5$ are commonly used, which for the illustration in Figure 2 are about 100%, 70%, and 30% respectively.

X_{max}

By its very nature, the largest event in the data-set is a property of a data-set under consideration. For this reason, I prefer the convention employed by Gibowicz and Kijko (1994), and Kijko and Funk (1994), who refer to this value as X_{max} (the maximum value of set X) although M_{obs} or M_{max}^{obs} is also used in literature.

The value of X_{max} is an indicator of hazard level in that it provides a lower bound of the largest event that can be expected in future ($M_{UL} > X_{max}$). Unless there is a significant and proven change in the conditions, the only defensible assumption is that an event larger than X_{max} can occur. This condition may be a significant change in the seismic regime that can be brought about by, for example, a change in the mining method or when mining moves from strong brittle ground into squeezing conditions.

X_{max} is, of course, highly dependent on the size and representativeness of the subset of data under consideration and X_{max} of the subset loses significance as a hazard indicator when a subset is temporally or spatially too small. X_{max} as a hazard indicator captures only the historical experience and does not in any way account for the stochastic nature of seismicity.

THE a/b -value

The value of a fitted GR relationship at $N = 1$ is sometimes also referred to as ' a/b ' (Hudyma, 2008) since it is equivalent to the ratio of the a and b parameters of the GR relationship (Equation [1]). It is important to note that a/b is a property of the fitted statistical model and not of the underlying data. The a/b -value is commonly used as an indicator of seismic hazard level with the meaning of 'the largest expected event' assigned to it.

It appears that this practice of interpreting a/b as the maximum event size is reinforced by the misunderstanding of the FM graph plotted for historical data (Figure 2). As the minimum value of the logarithmic y-axis is generally plotted as unity, and since no fractions of events can occur, this is interpreted as the 'end of the graph'. The GR relationship, however, is not a representation of the discrete events that occurred but a statistical best-fit model describing the relative frequency/probability of different sizes of event. There is no fundamental reason to stop the graph at $N = 1$. Interpreting the reverse cumulative distribution to terminate at $N = 1$ ignores the upper tail of the distribution, similar to the red line illustrated in Figure 3.

From the TGR distribution with the number of events determined by a , the probability of exceeding a/b is given by the following equation derived in Appendix A and plotted in Figure 4:

$$P \left[X_{max} \geq \frac{a}{b} \right] = 1 - \left(\frac{1 - 10^{-a}}{1 - 10^{-a \cdot k}} \right)^{10^a} \quad [9]$$

Which, for the open-ended GR relationship, reduces to the following:

$$P \left[X_{max} \geq \frac{a}{b} \right] = 1 - (1 - 10^{-a})^{10^a} \quad [10]$$

where:

X_{max} is the largest experienced event

b is the coefficient quantifying the log relative occurrence of smaller to larger events

a is the coefficient quantifying the number of events

$N @ 0 = 10^a$

k is the Fraction by which M_{UL} is larger than a/b ,

i.e. $M_{UL} = k \cdot a/b$

For an open-ended GR relationship, there is a > 63% chance of exceeding a/b . The probability is less for the truncated distribution. For situations where M_{UL} is large compared to a/b , the probability of exceeding a/b is large and the probability of exceeding a/b reduces for a/b closer to M_{UL} . The probability of

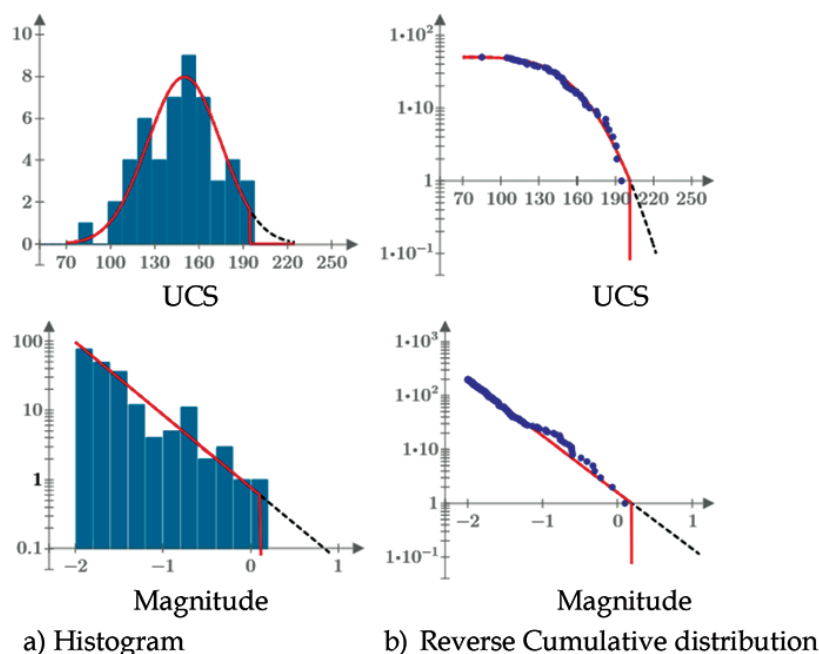


Figure 3—Truncation of statistical distribution at $N = 1$ to the largest point in the data-set for both UCS data and seismic magnitude data

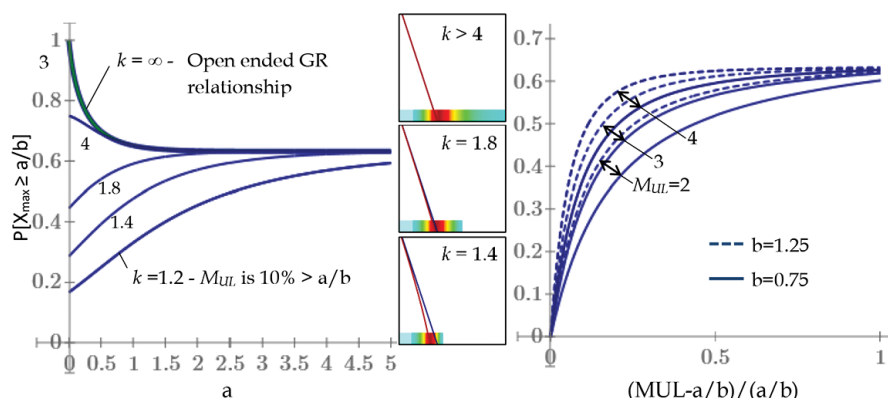


Figure 4—Probability for exceeding a/b

exceeding a/b where $a/b \geq M_{UL}$, per definition, zero. Note that these statements refer to the probability of exceeding a/b within the number of events determined by the a , i.e. $N @ 0 = 10^a$.

The a/b -value is an important parameter and it has a very clear meaning not generally appreciated. The a/b -value is the mode of the distribution of the largest event (the mode of f_{max}) (Figure 5). This is true for both the GR and TGR relationships (Appendix A).

The a/b -value is often used without consideration of spatial or temporal normalization, in which case it is a function of the subset of data under consideration and it loses any meaning as a hazard indicator. This will be discussed in greater detail further in the paper.

M_{max} and M_{UL}

In physical terms, M_{max} is used to define the region-characteristic maximum possible event magnitude, or, as the upper limit of event magnitude for a given region (Kijko and Singh, 2011). In other words, the largest magnitude that physical conditions will allow. In crustal seismology, this value is generally assumed to be constant for a particular seismic source zone. However, in

the mining environment, the value of M_{max} is not constant and is influenced by a number of factors, for example, rock mass conditions, mining-induced stress state, the mining sequence, and mining layout. In addition, M_{max} is expected to increase with the extraction ratio (Mendecki, 2012).

As mentioned before M_{max} and M_{UL} , are mostly used interchangeably. Conceptually M_{max} has the previously mentioned physical meaning, whilst M_{UL} carries only the abstract meaning of being an upper limit of a probability distribution. I find it useful to keep these two concepts separate as one may choose to use a large value for M_{UL} , say M6, without implying that $M_{max} = 6$. The only implicit statement is that $M_{max} < 6$. Due to several difficulties which will be discussed in further detail in the following paragraphs, estimates of M_{max} is subject to a great deal of uncertainty. For the purpose of hazard assessment and management, however, choosing a conservative but realistic value for M_{UL} will suffice.

When assessing the probability of exceeding a specified magnitude $P[M > M_L]$, underestimation of M_{UL} with an error of δ ($M_{UL} = M_{max} - \delta$) leads to larger errors than overestimation of M_{UL} by the same amount ($M_{UL} = M_{max} + \delta$). Underestimating M_{UL} is always optimistic, whilst overestimation is always conservative (Wesseloo, 2018). For the purpose of hazard assessment it is, therefore, prudent to use values for M_{UL} that are deliberately conservative.

Estimating M_{max} and M_{UL}

Wesseloo (2018) suggested the use of several methods discussed by Kijko (2004) to calculate M_{max} plus the associated standard deviation, and assigning the maximum of these values to M_{UL} .

$$M_{UL} = \max(M_{max\ i} + \Delta_i) \quad [11]$$

where:

M_{UL} is a conservative but realistic upper limit for f_{max} .

$M_{max\ i}$ is the largest magnitude that physical conditions will allow estimated with method i .

Δ_i is the standard deviation of the estimation of $M_{max\ i}$

The reliable and robust estimate of M_{max} is not a trivial task and several researchers have invested considerable effort in finding reliable estimates from seismic event catalogues, with most of the effort directed at application in crustal seismology (e.g. Kijko and Funk, 1994; Kijko, 2004; Lasocki and Urban, 2011; Kijko, 2012). These methods, based on record statistics, aim to estimate the largest possible value based on the recorded data.

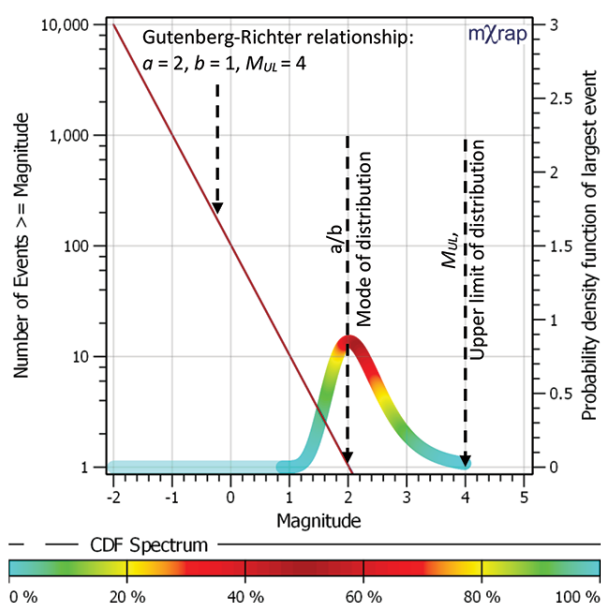


Figure 5—The a/b -value as the mode of distribution of the largest events within a given number of events

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The simplest and most well-known method for estimation of M_{UL} is the Robson–Whitlock method formulated as follows (Kijko, 2004):

$$M_{max} = X_{max} + (X_{max} - X_{max-1}) \quad [12]$$

where

X_{max} and X_{max-1} are the largest and second-largest recorded events

Among the other more common methods discussed by Kijko (2004) and (Kijko and Singh, 2011) are the Tate–Pisarenko, Kijko–Sellevoll, Order statistics, and Robson–Whitlock–Cooke and Cooke 1980.

It is important to note that the reliability of the estimate of M_{max} is highly dependent on the number of events on which this assessment is based. This dependence is investigated using Monte Carlo analysis on synthetically generated data-sets sampled from a specified FM distribution. For this analysis, $M_{max} = 2.5, 3.5$ and 4.5 , $b = 1$, and a -values from 1 to 4.5 ($N@0 = 10$ to 32 000) were used. The results for the Robson–Whitlock method are shown in Figure 6. The value for M_{max} is largely underestimated, with an underestimation of more 80% for an $a/b < M_{max} - 1$.

It is clear from the analysis that reasonably accurate estimates for M_{max} can only be obtained from large data-sets ($N@0 > 1000$ for $M_{max} = 2.5$ $N@0 > 10\ 000$ for $M_{max} = 4.5$). This implies that when the FM relationship is evaluated for smaller data-sets, the estimation M_{max} should not be limited to that small subset of data.

Artificial Black Swan events

The effect of variability

According to the Financial Times Lexicon, a Black Swan

event is 'An event or occurrence that deviates beyond what is normally expected of a situation and that would be extremely difficult to predict.' Since the M_{UL} is the upper truncation of the FM distribution, any event with magnitude greater than M_{UL} is, per definition, assigned a zero probability of occurrence. Underestimation of M_{UL} , ($M_{UL} < \text{the true } M_{max}$), will assign a zero probability to magnitudes that could reasonably be expected, should the $M_{UL} > \text{the true value of } M_{max}$.

Consider the following example based on synthetic magnitude distribution data. Using random deviate sampling, magnitude values were sampled from a TGR relationship with the following properties: $M_{UL} = 4$, $m_{min} = -2$, $b = 1$, and $N@-2$ is 100 000, i.e. $a = 3$. The results of several different approaches for estimating M_{UL} are shown in Figure 7.

Each of the sub-figures shows the FM distribution data; the best-fit GR (blue) and TGR (red) relationships. The probability distribution of the largest event for the given TGR relationship is shown as a spectrum at $N = 1$. The only differences between the sub-figures are the M_{UL} value and the resulting changes in the TGR relationship and the distribution of the largest events, f_{max} , derived from it.

In Figure 7a, the value of $M_{UL} = M_{max}$ obtained with the Robson–Whitlock method (Equation [12]). In Figure 7b, M_{UL} is taken as the maximum of all of the methods listed in the previous section. In Figure 7c, the M_{UL} is taken as the maximum of all the aforementioned methods with an added standard deviation for each calculation, according to Kijko and Singh (2011), assuming a magnitude resolution of 0.1. Finally, in Figure 7d, $M_{UL} = \text{the true } M_{max} = 4$ is used. Even with M_{UL} calculated as the maximum of the mentioned methods (Figure 7b) or with Equation [11], the probability of $M > 3.5$ is zero and 12% respectively whilst the actual value is 20%.

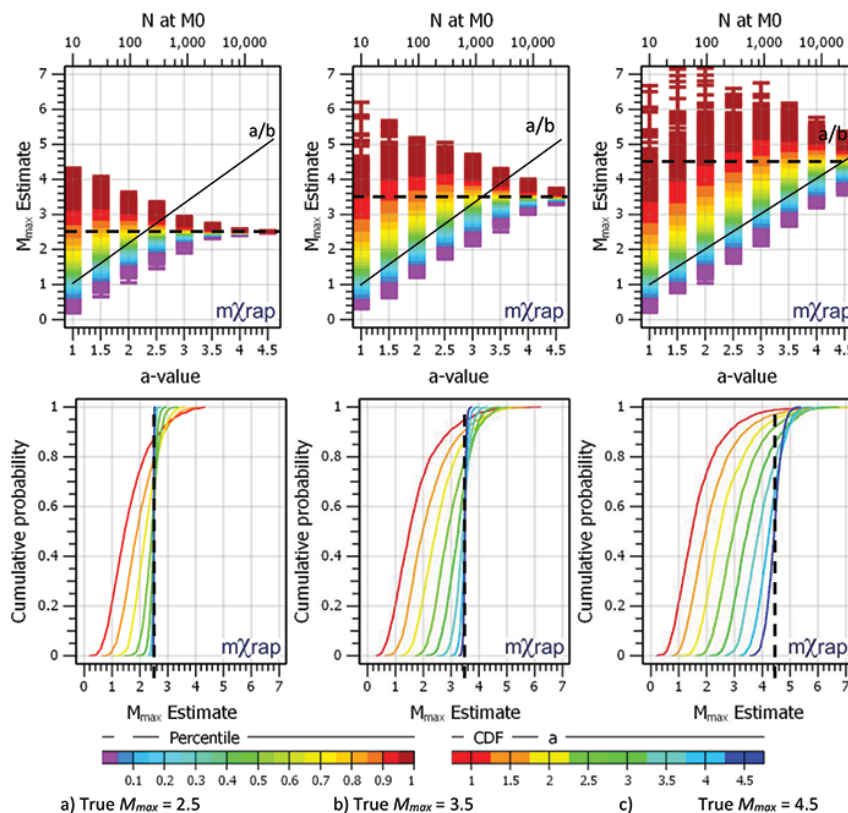


Figure 6—Uncertainty in the estimation of M_{max} using the Robson–Whitlock method for different a -values, $b = 1$ and a true M_{max} of (a) 2.5 (b) 3.5 and 4.5

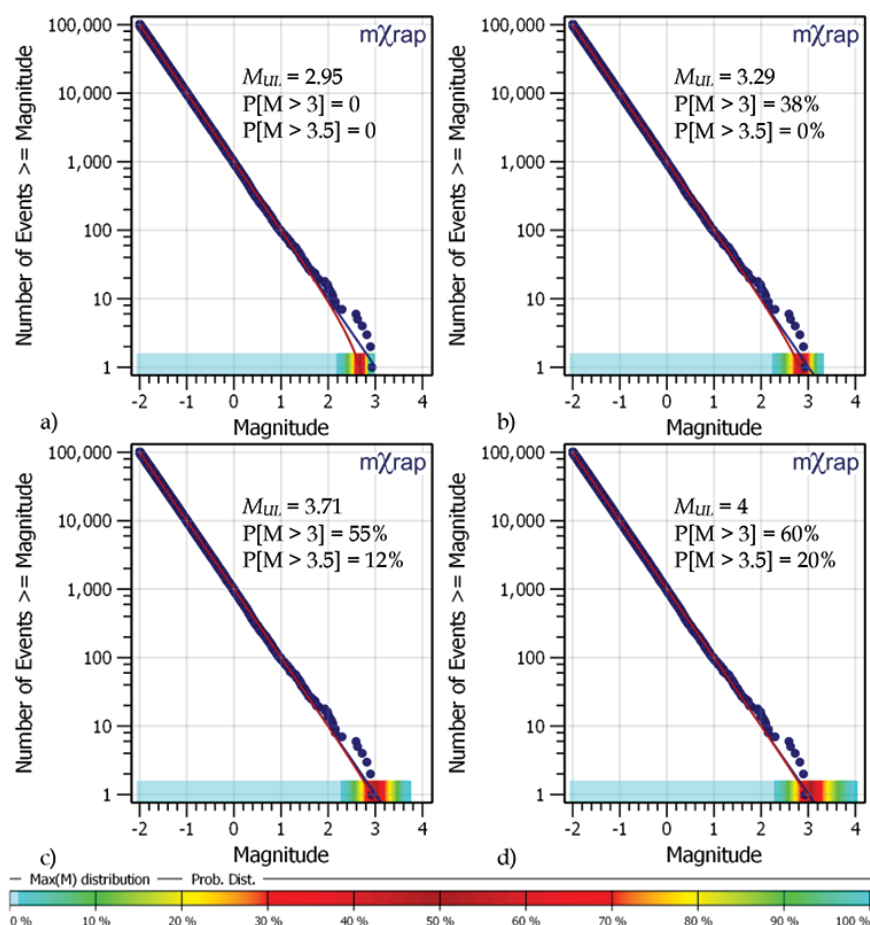


Figure 7—Comparison to probabilistic evaluations resulting from different values of M_{UL} taken as (a) the value Robson–Whitlock estimate. (b) the maximum of Tate–Pisarenko, Kijko–Sellevoll, Order statistics, Robson–Whitlock–Cooke and Cooke 1980 methods. (c) the maximum of all the aforementioned methods with an added standard deviation for each calculation assuming a magnitude resolution of 0.1. (d) the true value of $M_{max} = 4$

The effect of sensor limitations

Apart from the underestimation of M_{max} that may occur simply due to variability in the data, underestimation may also result from limitations in the lower frequency limit of sensors. Morkel and Wesseloo (2017) described the problem occurring when the lower frequency limit of the sensors is not sufficiently low to record the low frequency content of large events and no correction for this effect is made. When this occurs, it will lead to the under-recording of the moment (and potency). Magnitude scales dependent on moment are thus susceptible to the underestimation of large magnitudes. This includes moment magnitude and magnitude scales defined as a function of both energy and magnitude. Since the estimation of energy is not sensitive to the under-recording of low frequencies (Boore, 1986), this effect is less pronounced for magnitude scales based on both energy and moment. Mendecki (2013a) performed a quick survey of 100 mines using IMS systems and found that 25% used a moment-based magnitude scale while the rest used a scale defined as a function of both energy and moment. For simplicity, the following discussion is limited to moment magnitude.

As a result of the under-recording of the low-frequency content and subsequent underestimation of the moment magnitude, the FM distribution of the recorded data exhibits a nonlinear (on the log-linear scale) relationship, as shown in Figure 8.

Figure 8a shows the FM distribution of recorded data from a mine network with only 50 Hz sensors. The downward curvature of the distribution is not a result of the underlying statistical behaviour of moment magnitude, but of the under-recording of the moment by the sensors. Also shown in the figure are two theoretical lines: the straight GR relationship assumed as the true distribution of moment magnitude; and a theoretical assessment of the effect of under-recording using analytical formulations (Boore, 1986; Di Bona and Rovelli, 1988; Mendecki, 2013a; Morkel and Wesseloo, 2017). The effect of different lower frequency limits of the sensors is illustrated in Figure 8b. The amount of under-recording for different moment magnitudes and lower frequency limits are shown in Figure 9.

For databases subjected to under-recording, underestimation of M_{max} will result from the use of the statistical methods. To illustrate this problem, consider the following scenario plotted for $N@0 = 300$ in Figure 10. A seismic system with sensor lower limits of 14 Hz recording events from a seismic source with $b = 1$, $M_{max} = 3$. Figure 10 shows the assumed FM distribution as a blue line, and the synthetic events sampled from that distribution as light grey. The dark blue points are adjusted for the frequency limit of the sensors, according to Boore (1986). The calculated value of M_{max} in this case, is 2.53 which is smaller than the actual value of the largest experienced event on which the assessment is based, $M_{2.68}$. The red line shows the truncated GR relationship based on the recorded events.

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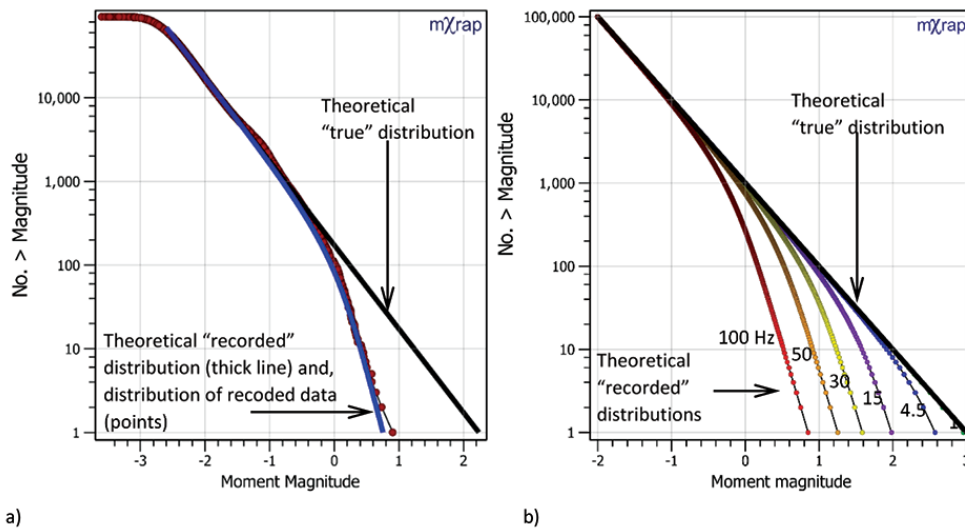


Figure 8—The effect of sensor frequency limits on the FM distribution of recorded seismic datasets, a) database and theoretical estimation of a 50 Hz system, b) theoretical estimation of moment magnitude under-recording for different lower frequency limits (Morkel and Wesseloo, 2017)

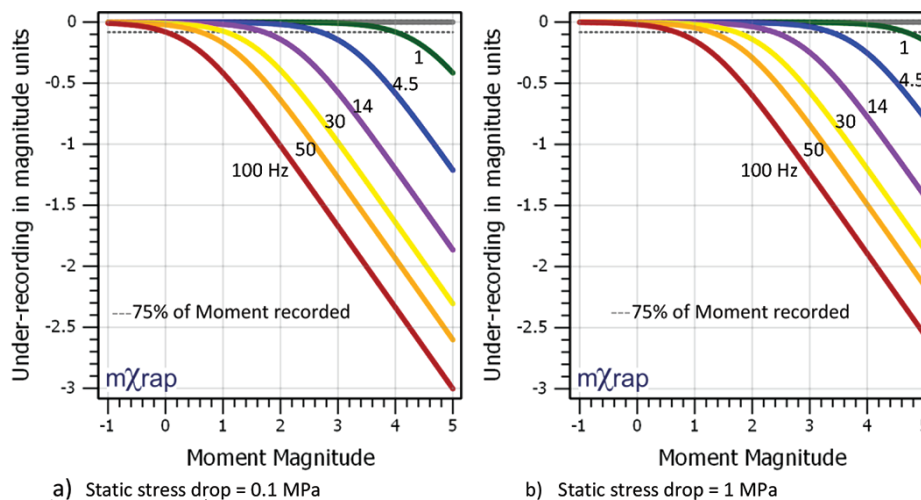


Figure 9—Under-recording of moment magnitude for different magnitudes and lower frequency limits

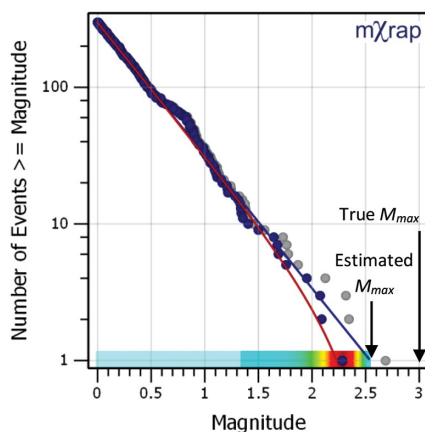


Figure 10—Synthetic data with adjustments for frequency under-recording, lower frequency = 14 Hz and a static stress drop of 0.1 MPa

Similar responses also occur for other frequency limits but the magnitude range at which the deviation becomes significant, differs. The lower the frequency limit of the sensor, the larger the magnitude that will be adequately recorded.

To further quantify the effect of the lower frequency of sensors on the estimation of M_{max} , a Monte Carlo analysis was performed, similar to that for which the results are shown in Figure 6. For this analysis, a lower frequency limit of 14 Hz and a static stress drop of 1 MPa are assumed. Excluding the added effect of the lower frequency of the sensor, these two analyses are the same. The results of this analysis are shown in Figure 11. The two sub-figures in column (a) display results for $M_{max} = 2.5$, (b) for $M_{max} = 3.5$, and (c) for $M_{max} = 2.5$. The top figure in each case presents the results for the Robson-Whitlock method. The results shown in the bottom figure in each case are obtained with Equation [11] using the Tate-Pisarenko, Kijko-Sellevoll, Order statistics, Robson-Whitlock-Cooke and Cooke 1980 methods.

It is clear from the comparison between Figure 6 and Figure 11 that the under-recording of moment due to sensor limitation can result in errors in the estimation of M_{max} , which are always optimistic and can be significant.

The best solution to this problem is to include lower frequency sensors in the system which are able to adequately record the lower frequency content of the event sizes expected at

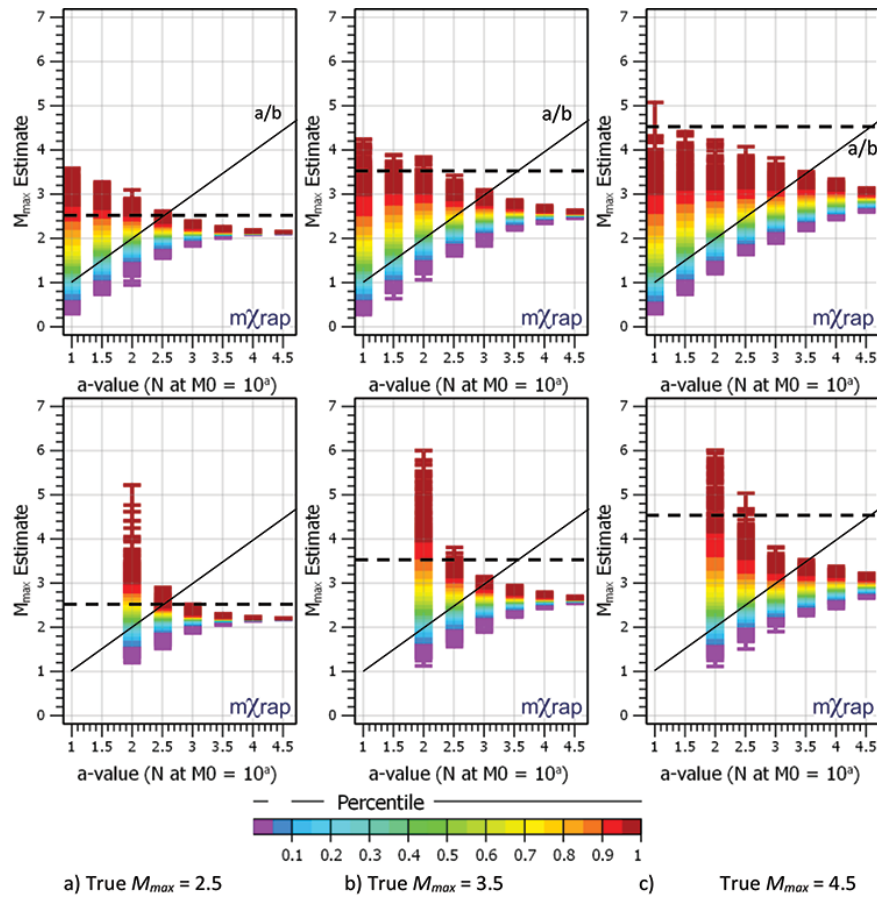


Figure 11—The effect of lower frequency limit on the estimation of M_{max} . Lower frequency = 14 Hz, $b=1$, static stress drop = 1 MPa. The top three figures present the results for the Robson-Whitlock method. The bottom three figures show the results from Equation [11] using the Tate-Pisarenko, Kijko-Sellevoll, Order statistics, Robson-Whitlock-Cooke and Cooke 1980 methods

the mine (Figure 9). In lieu of this, corrections may be applied to the recorded values to compensate for the effect of the sensor under-recording (Morkel and Wesseloo, 2017).

Spatial distribution of seismic hazard

Probabilistic hazard calculation is commonly performed on data within spatial filters. Such spatial filters are often delineated with respect to mining infrastructure, often with arbitrary size. Such arbitrarily chosen volumes can have a significant influence on the assessment and may influence decision-making.

The sensitivity of hazard assessment to arbitrarily chosen volumes relates to the spatial distribution of b -value, the spatial distribution of events, and the difference in volume for these arbitrary spatial filters.

The influence of the spatial distribution of b -value

It is important to note that, when different seismic sources with different b -values are lumped together and the hazard calculated, the total hazard will be different from that when calculating the total hazard based on the separate sources.

To illustrate this point, consider the idealized case illustrated in Figure 12 with a square subdivided into equal sub-areas on the left-hand side. On the right-hand side, the whole area is evaluated as a single unit. By way of analogy, these squares represent a mining area. For this illustration, we would like to answer the following question:

What is the probability of $M \geq 2$?

For the purpose of this illustration, we will assume a constant event rate.

An approach often used in mining is to evaluate the whole area, A , (for example a large mining block or corridor) obtaining the a - and b -values for the whole area, and calculate the required probability value using the following equation.

$$P(M \geq m)_A = 1 - F_{max}(a_A, b_A, m) \quad [13]$$

where

m is a specified threshold magnitude

An alternative approach would be to calculate the a - and b -values for each of the sub-areas and calculate the overall probability as follows:

The b -value for the combined area can be obtained from that of the sub-volumes as follows:

$$P(M \geq m)_i = 1 - F_{max}(a_i, b_i, m) \quad [14]$$

$$P(M \geq m)_A = 1 - \prod_i 1 - P(M \geq m)_i \quad [15]$$

$$b = \frac{\sum_i 10^{a_i}}{\sum_i \frac{10^{a_i}}{b_i}} \quad [16]$$

where:

a_i, b_i = a - and b -values for sub-volume i

It can be shown that for cases where the b -value over area A is constant, *i.e.* where $b_i = b_A$, the two approaches yield the same

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Calculate for the whole area			Calculate sub-areas and integrate the result			
n_A	b_A	$P(ML > 2)_A$	n_i	b_i	$P(ML > 2)_i$	$P(ML > 2)_A$
45	1.0	0.63	15	0.75	0.618	0.72
			5	1	0.119	
			10	1.2	0.114	
			15	1.5	0.058	

Figure 12—Illustration of the difference between evaluating the seismic hazard as a single unit and for smaller units with constant b -value

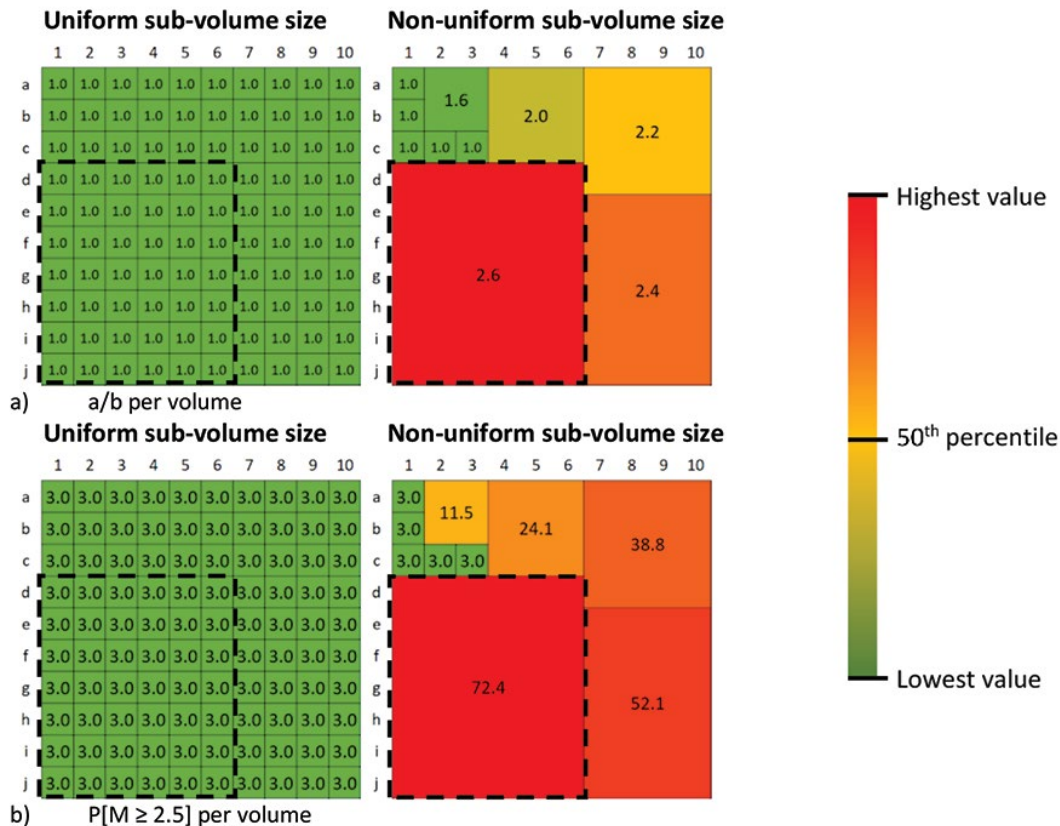


Figure 13—The influence of volume size on the evaluation of hazard (a) a/b distribution for uniform volume size and (b) probability of exceeding M2.5, $M_{UL} = 4$, $m_m n = 0$

result. However, where this is not the case, the first approach does not yield the correct answer. For the example, in Figure 12 the difference between the two approaches leads to a difference of 9% probability of exceeding M2.

This illustration shows that the probabilistic evaluation representative of the volume under consideration can only be achieved by integration of the results obtained for each sub-volume where the sub-volumes are small enough to represent a volume with constant b -value.

The influence of the volume of spatial filters

For a given b - and M_{UL} value, the number of events within a spatial filter will determine the seismic hazard for that spatial filter. The effect of hazard quantification and comparison between arbitrarily chosen spatial filter volumes without volume normalization is illustrated in Figure 13 and Figure 14. The square area shown on the left-hand side in Figure 13 and Figure 14 represents a whole mining area. In the left-hand side figures, the whole area consists of a hundred equally sized sub-areas which, for this example, each has a uniform distribution of events. The right-hand side figures display the same scenarios as that of the left-hand side figures, except that in these cases the

whole mining area is subdivided into arbitrarily chosen spatial filter areas.

In Figure 13, the whole mine has the same seismic event rate of $N@0 = 10$ and $b = 1$. As a result, the spatial distribution of hazard is uniform throughout the whole mine with an $a/b = 1$ (Figure 13a left), and a probability of exceeding M2.5 at 3% *per small square* (Figure 13b left). The a/b -values and the probability of exceeding M2.5 are shown in the right-hand side figure for each of the arbitrary spatial filter volumes. The highlighted sub-area consists of 42 small areas and therefore has $N@0 = 420$. The TGR (red line), $a/b, f_{max}$ (coloured distribution) for a small square and the highlighted area are shown in Figure 15.

Evaluating the seismic hazard for arbitrary volumes leads to the amplification of the hazard for larger volumes and a misrepresentation of the hazard. Figure 14 illustrates the same effect in a different scenario where the event rate is not uniformly distributed throughout the whole mining area. Figure 14 illustrates the fact that arbitrarily chosen filter volumes can mask the true spatial distribution of seismic hazard. These comparative hazard maps can be corrected by normalizing the assessment with respect to the volume.

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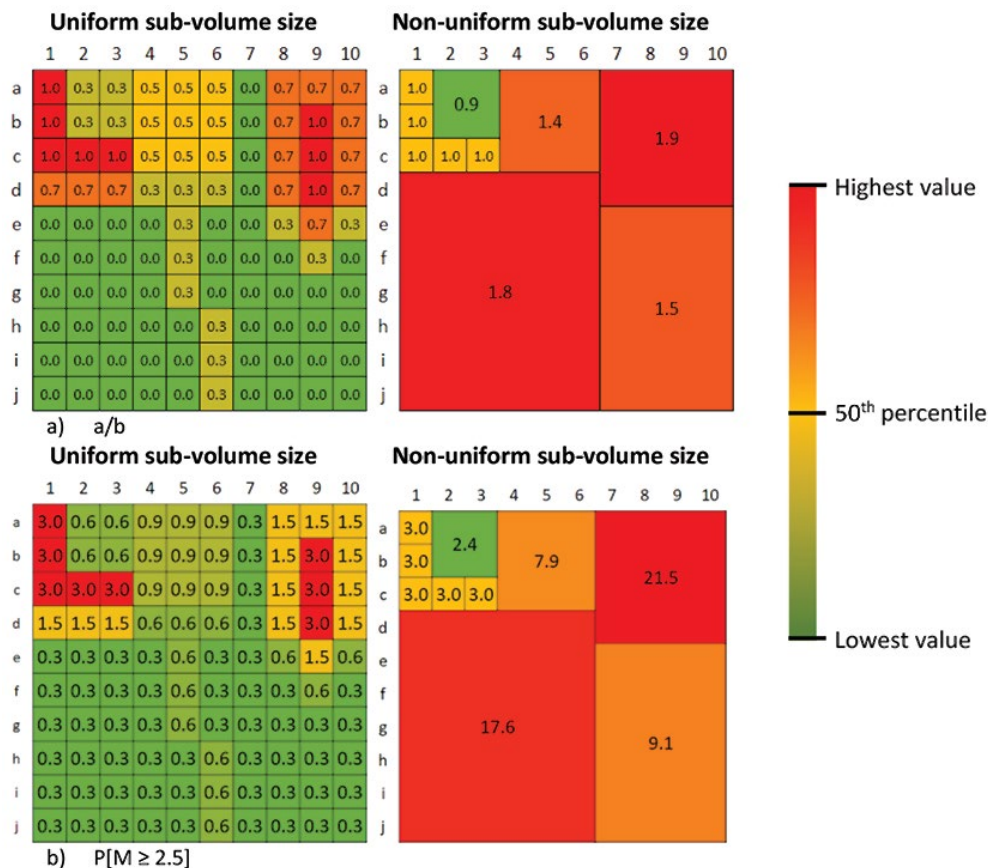


Figure 14—The influence of volume size on the evaluation of hazard (a) a/b distribution and (b) probability of exceeding $M_{2.5}$, $M_{UL} = 4$, $m_{min} = 0$

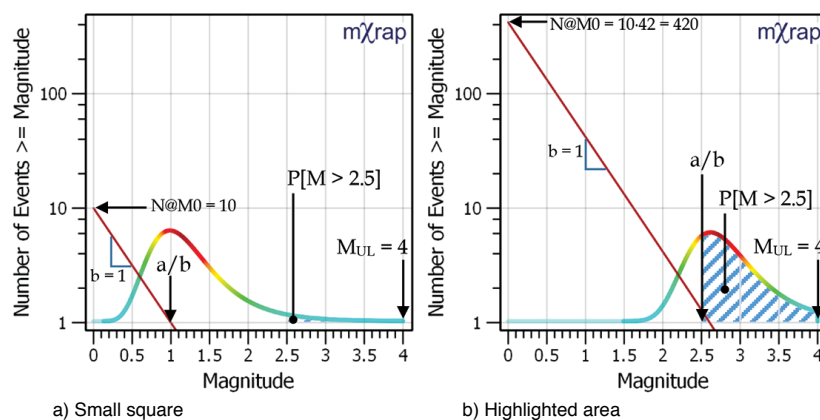


Figure 15—The TGR (red line), a/b , and f_{max} (coloured distribution) for the (a) small squares, and (b) the highlighted area in Figure 13

Hazard normalization

Spatial normalization

For the case illustrated in Figure 13, the statement was made that the probability of exceeding $M_{2.5}$ per small square is 3%. In this statement, the hazard is normalized with respect to volume. This normalized value is essential for obtaining a spatial distribution of hazard or a true comparison of hazard between different volumes but it does not provide the absolute hazard. The hazard for the whole mining area is not 3%. For the whole mining area, we need to integrate the hazard of each of the sub-areas to obtain 95%. In other words, there is about a 95% chance of experiencing $M_{2.5}$ anywhere in the mine, but it is equally likely to occur

anywhere in the mine with a 3% probability of occurring in any one of the small squares.

The total hazard for the whole mine and the spatial distribution of that hazard are independent of the size of the small squares, but the actual value associated with the small square is dependent on the size of the squares. To represent the spatial distribution in a way that is independent of the sub-square size, one would need to define a characteristic volume to which all values are normalized. It should be noted that the size of a characteristic volume does not influence the hazard calculations nor the spatial hazard distribution, but only influences the actual numbers by which the hazard is expressed. Wesseloo (2018) suggested the use of a volume size equal to that

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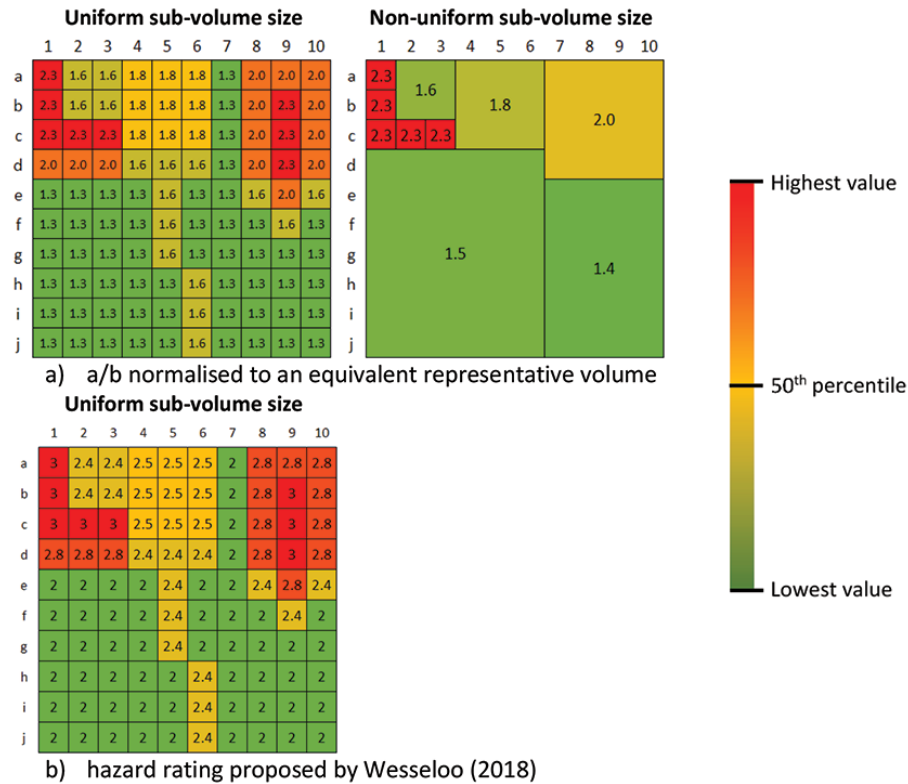


Figure 16—Hazard rating based equivalent representative volume, independent of the sub-square size

of a sphere with a radius of 50 m. The a/b -values in the scenario shown in Figure 14 are spatially normalized to a characteristic volume shown in Figure 16a.

The sub-square size of 20 m was assumed, and since the example is a two-dimensional one, an equivalent representative area with radius 50 m was used. The comparison between the non-uniform sub-volume scenarios in Figure 14a and Figure 16a shows that the normalization to a characteristic volume enables a more reasonable comparison of seismic hazard distribution. Spatial normalization, however, does not correct for the loss of information that occurs when the spatial filters do not take account of the underlying seismic sources, as only a mean value for each sub-volume is retained. As a result, some masking of seismic hazard trends still occur. The left-hand side diagram of Figure 16a illustrates the fact that when the evaluation volume is small enough to capture the change in seismic sources in space, the normalized a/b values provide a useful hazard rating for quantifying the spatial distribution of the seismic hazard.

Wesseloo (2018) takes this one step further and defines a hazard rating based on the same definition for the characteristic volume. The hazard rating is defined as the magnitude with a 15% probability of exceedance within the equivalent representative volume. This hazard rating definition was chosen to produce similar rating values to the hazard scale originally proposed by Hudyma and Potvin (2004), with which many mines in Australia and Canada are familiar. For the scenario in Figure 14b, this leads to the spatial hazard rating shown in Figure 16b. This approach provides a method for representing the spatial distribution of hazard which is independent of the size of the sub-volume.

Time normalization

The normalization of hazard is as important in the time domain

as it is in space. To enable direct comparison between hazards of different durations, it is necessary to normalize the calculated probability to the same equivalent timescale. This normalization can be performed as follows (Wesseloo, 2018):

$$P_{Tn} = 1 - (1 - P_{Te})^{\frac{Tn}{Te}} \quad [17]$$

where

Tn, Te are the normalized timeframe and the original timeframe, respectively

P_{Tn}, P_{Te} are the equivalent probabilities expressed for timeframes Tn and Te , respectively

A hazard with a weekly probability of 1% can be expressed with equivalent annualized values as $1 - (1 - 0.01)^{52} = 40\%$, while a hazard with a biennial probability of 50% can be expressed in equivalent annualized values as $1 - (1 - 0.5)^{0.5} = 30\%$.

Normalization can be performed to any timescale but short time periods should be avoided as this leads to small numbers which are often misinterpreted as small hazards. The use of one year (annualization) seems a reasonable approach and corresponds with the practice in other branches of engineering, financial risk management and corporate governance (Jonkman, van Gelder, and Vrijling, 2003; Terbrugge *et al.*, 2006; Stacey, Terbrugge, and Wesseloo, 2007; Wesseloo and Read, 2009). Annualization also allows one to calculate the associated risk and evaluate it against corporate accepted annualized risk levels.

Annualization of hazard values leads to an 'annual probability', but it is important to note that this should not be interpreted as the probability for a physical year (future or historical). It is the probability value appropriate to the timescale for which the mean seismic rate is applicable, expressed in equivalent annualized terms.

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The example calculations following Equation [17] assume that both hazards are present over a long time period. If, for example, a hazard with a weekly probability of 1% is present only for one week in the year, the yearly hazard would also be 1%. This sometimes leads to misunderstanding in the application of hazard assessments in industry where a short-lived but repeating hazard is sometimes evaluated in isolation.

Seismic hazard in a mine is transient in space and time and, although the seismic hazard at a specific location might be short-lived, the hazard is of a repeating nature. For example, the hazard associated with the mining of a single stope might be present only during the time it takes to mine that stope, but, a similar hazard might occur due to the mining of the next stope. Both long-term and repeating hazards can be normalized temporally using Equation [17].

Wesseloo (2018) illustrated this concept with the following fictitious mining scenario that consists of 12 stopes (Figure 17). Each of the stopes is mined for a month, during which a seismic response is induced in the indicated area around it. The seismicity in this surrounding area ceases when mining in this stope is complete. During the following month, the next stope is mined with the associated induced seismicity limited to its surrounding area. The argument can be further simplified by assuming a constant $b = 1$ over the whole volume and the whole year, and by assuming that the total number of events occurring in the surrounding area of each of the stopes is the same at $N@-2 = 1000$ ($a = 1$).

For each stope, the probability of exceeding M2 is 3.92%, and if evaluated in isolation, may be regarded as acceptable. Cumulating the number of events for all 12 stopes, however, results in the total probability of exceeding M2 of 38%. In the year, the company is exposed to the *total* aggregated hazard of $P[M > 2] = 38\%$, even though each stope only has an individual monthly hazard of probability 3.9%. If exposing the company to the yearly hazard of 38% is not acceptable, by implication, it is not acceptable to expose the company to the hazard associated with every one of those stopes individually.

Concluding remarks

The topic of seismic hazard assessment is subject to several misconceptions in the mining industry, and this paper addressed some of these. The term ' M_{max} ' is used for several different concepts which appear to contribute to these misconceptions. To avoid confusion, I suggest that the term ' M_{max} ' should be reserved for the concept of the maximum credible event. Other concepts referred to as ' M_{max} ' should be referred to by unique names. I propose the use of ' X_{max} ' or ' M_{obs} ' for the largest recorded event in a data-set, ' a/b ' for the value of the fitted GR relationship

at $N = 1$, and, ' M_{UL} ' for the upper truncation value of the FM distribution.

The value of the fitted GR relationship at $N = 1$ (a/b) is sometimes interpreted as the expected largest event. This value is, however, the *mode* of the distribution of the largest expected magnitude with a 63% chance of being exceeded for the open-ended GR relationship. The probability of exceedance is smaller for truncated GR relationship and depends on the upper limit of the distribution.

The accuracy of methods for estimating M_{max} is low when $a/b \ll M_{max}$ and tends to underestimate M_{max} . Reliable results can only be obtained when M_{max} is based on large data-sets, (e.g. $N@0 > 1000$ for $M_{max} = 2.5$ $N@M > 10\ 000$ for $M_{max} = 4.5$).

When assessing the probability of exceeding a specified magnitude $P[M > M_L]$, underestimation of M_{UL} ($M_{UL} = M_{max} - \delta$) leads to larger errors than overestimation of M_{UL} by the same amount ($M_{UL} = M_{max} + \delta$). Underestimating M_{UL} is always optimistic, while overestimation is always conservative. For the purpose of hazard assessment, it is therefore prudent to use values for M_{UL} that are deliberately conservative.

Artificial Black Swan events can be created when M_{UL} underestimates M_{max} . This can occur simply due to the effect of uncertainty. The under-recording of low frequencies due to sensor limitations leads to the underestimation of M_{max} and also leads to artificial Black Swan events. To combat this problem, sensors which are able to adequately record the lower frequency content of the event sizes expected at the mine, need to be included in the system. In lieu of this, corrections may be applied to the recorded values to compensate for the effect of the sensors under-recording.

Assessment of the seismic hazard needs to be performed on sub-volumes for which the b -value and the event rate can be assumed to be constant.

For comparative hazard evaluation in space and time, both spatial and temporal normalization are necessary. Normalization to a characteristic volume equal to a sphere with a 50 m radius is suggested. Annualization of hazards is proposed for temporal normalization.

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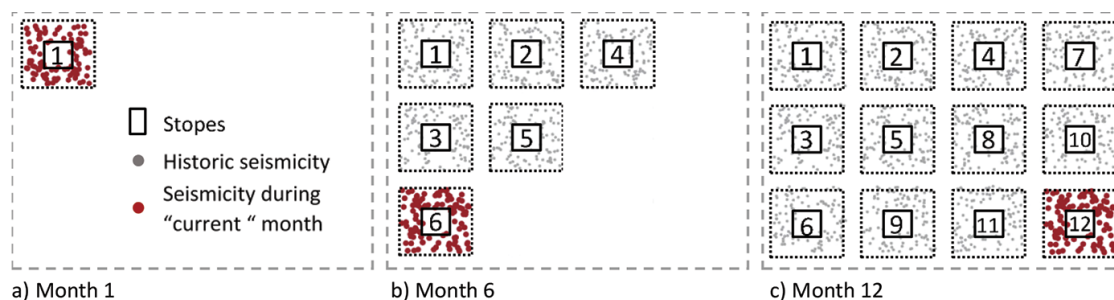


Figure 17 – Conceptual mine layout for an illustrative example, consisting of 12 stopes with a surrounding area of seismicity induced by the stoping activity (Wesseloo, 2018)

for proofing my language. The content of this paper has flowed from work done over the span of several years with the support of the mXrap Consortium.

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Appendix A

The probability of exceeding an event size of a/b within n events

For the TGR relationship, the CDF of the event distribution from the magnitude of completeness m_{min} is then given by

$$F(M) = 1 - \frac{1 - 10^{-b \cdot (M - m_{min})}}{1 - 10^{-b \cdot (M_{UL} - m_{min})}} \quad [A.1]$$

The probability of exceeding an event size of M within n events is given by (Gibowicz and Kijko, 1994):

$$F_{max}(M) = (F(M))^n \quad [A.2]$$

where:

$F_{max}(M)$ is the cumulative distribution function of the magnitude of the largest event

$F(M)$ is the cumulative probability density function describing the magnitude distribution of event size.

The probability of exceeding the value of a/b within n events of magnitude $\geq m_{min}$ is given by

$$P[X_{max} \geq \frac{a}{b}] = 1 - F_{max}(\frac{a}{b}) = 1 - F(\frac{a}{b})^n \quad [A.3]$$

And defining $M_{UL} = k \cdot a/b$ leads to

$$= 1 - \left(\frac{1 - 10^{-b \cdot (\frac{a}{b} - m_{min})}}{1 - 10^{-b \cdot (k \cdot \frac{a}{b} - m_{min})}} \right)^n = 1 - \left(1 - \frac{1}{n} \right)^n \quad [A.4]$$

This relationship is independent of m_{min} and can be written in terms of the a -value as

$$P[X_{max} \geq \frac{a}{b}] = 1 - \left(\frac{1 - 10^{-a}}{1 - 10^{-a \cdot k}} \right)^{10^a} \quad [A.5]$$

which, for the open-ended GR relationship reduces to

$$P[X_{max} \geq \frac{a}{b}] = 1 - (1 - 10^{-a})^{10^a} \quad [A.6]$$

The mode of the distribution of F_{max}

The mode of $f_{max}(x)$ can be obtained as follows:

$$\begin{aligned} \frac{d^2}{dx^2} F_{max}(x) &= \\ \frac{b^2 \cdot n \cdot \ln(10)^2 \cdot (A^{n-2} - n \cdot A^{n-2} + 10^{b \cdot x} \cdot A^{n-1})}{10^{2bx}} &= 0; \quad [A.7] \\ A &= (1 - 10^{-b \cdot x}) \end{aligned}$$

which reduces to $x = a/b$ for both the open-ended and TGR relationship. ♦